



Three positive solutions of nonhomogeneous semilinear elliptic equations

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Abstract

In this paper, we use new analyses to assert that there are three positive solutions of Eq. (1.1) in infinite cylinder domain with hole $\mathbf{A} \setminus \overline{D}$.

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1. Introduction

Throughout this paper, let $z = (x, y) \in \mathbb{R}^{N-m} \times \mathbb{R}^m = \mathbb{R}^N$, where $1 \leq m \leq N - 1$ and we denote $\omega \subset \mathbb{R}^{N-m}$ is a bounded smooth domain, the infinite cylinder domain $\mathbf{A} = \omega \times \mathbb{R}^m$; the infinite cylinder domain with hole $\mathbf{A} \setminus \overline{D}$, where $D \subsetneq \mathbf{A}$ is a bounded domain in \mathbb{R}^N which contained in $B^N(0; r_0) \cap \mathbf{A}$ for some $r_0 > 0$.

In this paper, we consider the multiplicity of positive solutions for semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + h(z) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \quad (1.1)$$

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where $2 < p < \frac{2N}{N-2}$ ($N \geq 3$), $\Omega = \mathbf{A} \setminus \overline{D}$ and $h(x, y) \in L^2(\Omega) \setminus \{0\}$. Associated with Eq. (1.1), we consider the functionals a , b , and J_h , for $u \in H_0^1(\Omega)$,

$$a(u) = \int_{\Omega} (|\nabla u|^2 + u^2), \quad b(u) = \int_{\Omega} |u|^p, \quad J_h(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u) - \int_{\Omega} hu.$$

By Rabinowitz [15, Proposition B.10], a , b , and J_h are of C^1 . It is well known that the solutions of Eq. (1.1) and the critical points of the energy functional J_h are the same. For the limiting case of Eq. (1.1): $h = 0$, we consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \quad (1.2)$$

It is known that the existence of positive solutions of the homogeneous equation (1.2) is affected by the shape of the domain. By the Rellich compactness theorem and the minimax method, it is easy to obtain a positive solution of Eq. (1.2) in bounded domains (see Ambrosetti–Rabinowitz [3]). For general unbounded domains Ω , because the lack of compactness, the existence of positive solutions of Eq. (1.2) in Ω is very difficult and unclear. The breakthrough was made by Esteban–Lions [11]. They asserted that Eq. (1.2) in Esteban–Lions domain does not admit any nontrivial solution. Recently, there have been some progresses for the existence of positive solutions of Eq. (1.2) in unbounded domains as follows: Benci–Cerami [4] for Ω an exterior domain, Berestycki–Lions [5] for $\Omega = \mathbb{R}^N$, Lien–Tzeng–Wang [14] for Ω an infinite cylinder domain \mathbf{A} , Chen–Wang [6] for Ω an interior flask domain, Del Pino–Felmer [9,10] for Ω a quasicylindrical domain, Wang [17] for Ω a Esteban–Lions domain with holes, Wu [19] for Ω a multi-bump domain.

In this paper, we are interested in the multiplicity of positive solutions for Eq. (1.1) in $\mathbf{A} \setminus \overline{D}$. Before stating our main results, we need the some notations: let λ_1 be the first eigenvalue of $-\Delta$ in ω with the Dirichlet problem and ϕ_1 the corresponding positive eigenfunction to λ_1 . Then we have the following result.

Theorem 1.1. *There exist positive numbers d_0, δ such that if $\|h\|_{L^2} < d_0$ and*

$$0 \leq h(x, y) \leq c \exp(-\sqrt{1 + \lambda_1 + \delta}|y|) \quad \text{for all } (x, y) \in \mathbf{A} \setminus \overline{D},$$

for some $c > 0$, then Eq. (1.1) in $\mathbf{A} \setminus \overline{D}$ has at least three positive solutions.

Our result generalizes previous results in two folds.

- (1) Hirano [12], Zhu [20], and Cao–Zhou [8] proved in \mathbb{R}^N and Hsu–Wang [13] in an exterior domain, that Eq. (1.1) admits two positive solutions. We generalize their results to obtain three positive solutions.
- (2) Consider the generalized equation of Eq. (1.1),

$$\begin{cases} -\Delta u + u = p(z)|u|^{p-2}u + h(z) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \quad (1.3)$$

where $0 < p(z) \leq 1$. Adachi–Tanaka [1] asserted that there are three positive solutions of Eq. (1.3) in \mathbb{R}^N . \mathbb{R}^N is contractible and there is a ground state solution in it. Our domain $\mathbf{A} \setminus \overline{D}$ is not contractible and there is no any ground state solution in it. So we need more analyses to work for it. We generalize the result of Adachi–Tanaka [1] to that $p(x) = 1$ and $\Omega = \mathbf{A} \setminus \overline{D}$.

Main tools are from Adachi–Tanaka [1], Tarantello [16], and Cao–Zhou [8]. We then develop analyses which include some lemmas for the limiting case $h = 0$ to complete our theory.

2. (PS)-theory

We define the Palais–Smale (denoted by (PS)) sequences and (PS)-conditions in $H_0^1(\Omega)$ for J_h as follows.

Definition 2.1.

- (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$ -sequence in $H_0^1(\Omega)$ for J_h if $J_h(u_n) = \beta + o(1)$ and $J'_h(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \rightarrow \infty$;
- (ii) J_h satisfies the $(PS)_\beta$ -condition in $H_0^1(\Omega)$ if every $(PS)_\beta$ -sequence in $H_0^1(\Omega)$ for J_h contains a convergent subsequence.

For the limiting case $h = 0$, we consider the Nehari minimization problem:

$$\alpha_0(\Omega) = \inf_{u \in \mathbf{M}_0} J_0(u),$$

where $\mathbf{M}_0 = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle J'_0(u), u \rangle = 0\}$. Note that a nonzero critical point $u \in H_0^1(\Omega)$ of J_0 is a ground state solution of Eq. (1.1) in Ω if $J_0(u) = \alpha_0(\Omega)$. Then we have the following results.

Lemma 2.2. *There is a bijective $C^{1,1}$ map m from the unit sphere Σ in $H_0^1(\Omega)$ to \mathbf{M}_0 . Moreover, \mathbf{M}_0 is path-connected and there exists a constant $c > 0$ such that for $u \in \mathbf{M}_0$, $\|u\|_{H^1} \geq c$ and $J_0(u) \geq c$.*

Proof. See Chen–Wang [6]. \square

Lemma 2.3. *Let $\beta > 0$ and $\{u_n\}$ be a sequence in $H_0^1(\Omega) \setminus \{0\}$ for J_0 such that $J_0(u_n) = \beta + o(1)$ and $a(u_n) = b(u_n) + o(1)$. Then there is a sequence $\{s_n\}$ in \mathbb{R}^+ such that $s_n = 1 + o(1)$, $\{s_n u_n\}$ in \mathbf{M}_0 and $J_0(s_n u_n) = \beta + o(1)$.*

Proof. See Chen–Wang [6]. \square

Lemma 2.4. *If $u \in H_0^1(\Omega) \setminus \{0\}$, then*

$$\left(\frac{a(u)^{\frac{p}{2}}}{b(u)} \right)^{\frac{1}{p-2}} \geq \left(\frac{2p}{p-2} \right)^{\frac{1}{2}} \alpha_0(\Omega)^{\frac{1}{2}}.$$

Proof. See Chen–Wang [6]. \square

Lemma 2.5. *$\{u_n\}$ is a $(PS)_{\alpha_0(\Omega)}$ -sequence in $H_0^1(\Omega)$ for J_0 if and only if $J_0(u_n) = \alpha_0(\Omega) + o(1)$ and $a(u_n) = b(u_n) + o(1)$. In particular, every minimizing sequence $\{u_n\}$ in \mathbf{M}_0 of $\alpha_0(\Omega)$ is a $(PS)_{\alpha_0(\Omega)}$ -sequence in $H_0^1(\Omega)$ for J_0 .*

Proof. See Wang–Wu [18]. \square

Associated with Eq. (1.2) in \mathbf{A} , we consider the functional J^∞ , for $u \in H_0^1(\mathbf{A})$,

$$J^\infty(u) = \frac{1}{2} \int_{\mathbf{A}} (|\nabla u|^2 + u^2) - \frac{1}{p} \int_{\mathbf{A}} |u|^p.$$

By Lien–Tzeng–Wang [14], Eq. (1.2) in \mathbf{A} has a positive solution $w(x)$ such that $J^\infty(w) = \alpha_0(\mathbf{A})$. Moreover, the positive solution $w(x)$ of Eq. (1.2) plays an important role in describing the asymptotic behavior of a (PS)-sequence for J_h .

Proposition 2.6. *Let $\{u_n\}$ be a (PS)-sequence in $H_0^1(\Omega)$ for J_h . Then there exist a subsequence $\{u_n\}$, an integer $k \in \mathbb{N} \cup \{0\}$, k sequences $\{z_n^1\}, \{z_n^2\}, \dots, \{z_n^k\} \subset \mathbf{A}$, a critical point $u_0 \in H_0^1(\Omega)$ of J_h and w^1, w^2, \dots, w^k are solutions of Eq. (1.2) in \mathbf{A} such that*

$$|z_n^i| \rightarrow \infty \quad \text{for } 1 \leq i \leq k,$$

$$u_n \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega),$$

$$u_n = u_0 + w^1(z - z_n^1) + w^2(z - z_n^1 - z_n^2) + w^k(z - z_n^1 - \dots - z_n^k) + o(1) \quad \text{in } H_0^1(\mathbf{A}),$$

$$J_h(u_n) = J_h(u_0) + \sum_{i=1}^k J^\infty(w^i) + o(1).$$

Proof. This is a standard result. See Lien–Tzeng–Wang [14] for analogous arguments. \square

Next, we give some properties of the functional J_0 .

Lemma 2.7. *We have*

$$(i) \quad \inf\{J_0(u) \mid u \in \mathbf{M}_0\} = \alpha_0(\Omega) = \alpha_0(\mathbf{A});$$

$$(ii) \quad \inf\{J_0(u) \mid u \in \mathbf{M}_0\} \text{ is not achieved.}$$

Proof. See Wang [17]. \square

Lemma 2.8. *There exists a $\delta_0 > 0$ such that if $u \in \mathbf{M}_0$ and $J_0(u) \leq \alpha_0(\mathbf{A}) + \delta_0$, then*

$$\int_{\mathbf{A}} \frac{y}{|y|} (|\nabla u|^2 + u^2) dy dx \neq 0.$$

Proof. On the contrary, there exists a sequence $\{u_n\}$ in \mathbf{M}_0 such that $J_0(u_n) = \alpha_0(\mathbf{A}) + o(1)$ and

$$\int_{\mathbf{A}} \frac{y}{|y|} (|\nabla u|^2 + u^2) dy dx = 0.$$

By Lemmas 2.5, 2.7, $\{u_n\}$ is a $(PS)_{\alpha_0(\mathbf{A})}$ -sequence in $H_0^1(\Omega)$ for J_0 . By Proposition 2.6 and Lemma 2.7, there exists a sequence $\{y_n\}$ in \mathbb{R}^m such that $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$ and

$$u_n(x, y) = w(x, y - y_n) + o(1) \quad \text{strongly in } H_0^1(\mathbf{A}).$$

Assume $\frac{y_n}{|y_n|} \rightarrow y_0$ as $n \rightarrow \infty$, where y_0 is a unit vector in \mathbb{R}^m . Then by the Lebesgue dominated theorem, we have

$$\begin{aligned}
0 &= \int_{\mathbf{A}} \frac{y}{|y|} (|\nabla u_n|^2 + u_n^2) dy dx \\
&= \int_{\mathbf{A}} \frac{y + y_n}{|y + y_n|} (|\nabla w|^2 + w^2) dy dx + o(1) \\
&= \left(\frac{2p}{p-2} \right) y_0 \alpha_0(\mathbf{A}) + o(1),
\end{aligned}$$

which is a contradiction. \square

3. Existence of a local minimum

In this section, we prove that there exists a positive solution of Eq. (1.1). First, we consider the Nehari manifold \mathbf{M}_h , where

$$\mathbf{M}_h = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle J'_h(u), u \rangle = 0\}.$$

Define $\psi(u) = \langle J'_h(u), u \rangle = a(u) - b(u) - \int_{\Omega} hu$. Then we have

Lemma 3.1. *Suppose that $h(z) \geq 0$ satisfies*

$$0 < \|h\|_{L^2} < (p-2) \left(\frac{1}{p-1} \right)^{\frac{p-1}{p-2}} \left(\frac{2p}{p-2} \right)^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}}.$$

Then for each $u \in \mathbf{M}_h$, we have $\langle \psi'(u), u \rangle = a(u) - (p-1)b(u) \neq 0$.

Proof. Our proof is almost the same as that in Tarantello [16]. \square

By Lemma 3.1, we write $\mathbf{M}_h = \mathbf{M}_h^+ \cup \mathbf{M}_h^-$, where

$$\mathbf{M}_h^+ = \{u \in \mathbf{M}_h \mid a(u) - (p-1)b(u) > 0\},$$

$$\mathbf{M}_h^- = \{u \in \mathbf{M}_h \mid a(u) - (p-1)b(u) < 0\},$$

and define

$$\alpha_h(\Omega) = \inf_{u \in \mathbf{M}_h} J_h(u), \quad \alpha_h^+(\Omega) = \inf_{u \in \mathbf{M}_h^+} J_h(u), \quad \alpha_h^-(\Omega) = \inf_{u \in \mathbf{M}_h^-} J_h(u).$$

For each $u \in H_0^1(\Omega) \setminus \{0\}$, we write

$$t_{\max} = t_{\max}(u) = \left(\frac{a(u)}{(p-1)b(u)} \right)^{\frac{1}{p-2}} > 0.$$

By elementary calculus, we have the following two lemmas.

Lemma 3.2. *For each $u \in H_0^1(\Omega) \setminus \{0\}$, we have the following results.*

- (i) *There is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-u \in \mathbf{M}_h^-$ and $J_h(t^-u) = \max_{t \geq t_{\max}} J_h(tu)$;*
- (ii) *$t^-(u)$ is a continuous function for nonzero u ;*

- (iii) $\mathbf{M}_h^- = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{H^1}} t^- (\frac{u}{\|u\|_{H^1}}) = 1\}$;
 (iv) If $\int_{\Omega} hu > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+u \in \mathbf{M}_h^+$ and $J_h(t^+u) = \min_{0 \leq t \leq t^-} J_h(tu)$.

Proof. Our proof is almost the same as that in Tarantello [16]. \square

Lemma 3.3.

- (i) For each $u \in \mathbf{M}_h^+$, we have $\int_{\Omega} hu > 0$ and $J_h(u) < 0$. In particular, $\alpha_h(\Omega) \leq \alpha_h^+(\Omega) < 0$;
 (ii) J_h is coercive and bounded below on \mathbf{M}_h .

Proof. Similar to the proof of Theorem 1 in Tarantello [16, p. 288]. \square

Proposition 3.4. J_h satisfies the $(PS)_{\beta}$ -condition for $\beta < \alpha_h(\Omega) + \alpha_0(\mathbf{A})$.

Proof. Similar to the proof of Corollary 1.10 in Adachi–Tanaka [1]. \square

Furthermore, we have the following theorem.

Theorem 3.5. Let $r_0 = (\frac{1}{p-1})^{\frac{1}{p-2}} (\frac{2p}{p-2})^{\frac{1}{2}} \alpha(\Omega)^{\frac{1}{2}}$. Then

- (i) $\mathbf{M}_h^+ \subset B(r_0) = \{u \in H_0^1(\Omega) \mid \|u\|_{H^1} < r_0\}$;
 (ii) There is a unique local minimum $u_0 \in \mathbf{M}_h^+$ of J_h such that $J_h(u_0) = \alpha_h^+(\Omega) = \alpha_h(\Omega)$;
 (iii) u_0 is a positive solution of Eq. (1.1).

Proof. Similar to the proof of same results in Adachi–Tanaka [1] and Cao–Zhou [8]. \square

Remark 3.1. Throughout this paper, let u_0 be the positive solution of Eq. (1.1) in Theorem 3.5.

4. Existence of three solutions

In this section, we assert that there are three positive solutions of Eq. (1.1) in $\Omega = \mathbf{A} \setminus \bar{D}$. By Lien–Tzeng–Wang [14], there is a positive ground state solution of Eq. (1.2) in \mathbf{A} . Let λ_1 be the first eigenvalue of $-\Delta$ in ω with the Dirichlet problem, and ϕ_1 the corresponding positive eigenfunction to λ_1 . Then we have the following results.

Theorem 4.1. Let w be a positive ground state solution of Eq. (1.2) in \mathbf{A} . Then for each $0 < \delta < 1 + \lambda_1$ there exist $\gamma > 0$ and $\beta > 0$ such that

$$\gamma \phi_1(x) \exp(-\sqrt{1 + \lambda_1 + \delta}|y|) \leq u(z) \leq \beta \phi_1(x) \exp(-\sqrt{1 + \lambda_1 - \delta}|y|)$$

for $z = (x, y) \in \mathbf{A}$.

Proof. See Chen–Chen–Wang [7]. \square

Lemma 4.2. Let u be a positive solution of Eq. (1.1) in Ω . Then for any $0 < \delta < 1 + \lambda_1$, there exist positive constants γ_1 , γ_2 and $R_0 > r_0$ such that for $|y| \geq R_0$,

$$\gamma_1 \phi_1(x) \exp(-\sqrt{1 + \lambda_1 + \delta}|y|) \leq u(x, y) \leq \gamma_2 \phi_1(x) \exp(-\sqrt{1 + \lambda_1 - \delta}|y|).$$

Proof. By the regularity results, we have $u \in W^{2,s}(\Omega) \cap C^{1,\theta}(\overline{\Omega})$ for some $0 < \theta < 1$ and $u(z) \rightarrow 0$ as $|y| \rightarrow \infty$.

(i) Take $R_1 \geq r_0$ such that $D \subset B^N(0; R_1) \cap \mathbf{A}$. For $0 < \delta < \min\{\varepsilon, 1\}$, we choose $R_2 > R_1$ such that

$$\delta - \frac{\sqrt{1 + \lambda_1 + \delta}(m-1)}{|y|} \geq 0 \quad \text{for } |y| \geq R_2. \quad (4.1)$$

Define $v_1(x, y) = \phi_1(x) \exp(-\sqrt{1 + \lambda_1 + \delta}(|y| - R_2))$. Let

$$\gamma_1 = \inf_{\substack{z \in \Omega \\ |y|=R_2}} \frac{u(x, y)}{v_1(x, y)},$$

similarly to the proof in Chen–Chen–Wang [7], $\gamma_1 > 0$. Then $\min_{|y|=R_2} (u - \gamma_1 v_1)(x, y) \geq 0$. By (4.1), for $|y| > R_2$,

$$\begin{aligned} \Delta(u - \gamma_1 v_1)(x, y) &= u - |u|^{p-2}u - h(x, y) - \left(\beta^2 - \frac{\beta(m-1)}{|y|}\right) \gamma_1 v_1 \\ &\leq u - \left(\beta^2 - \frac{\beta(m-1)}{|y|}\right) \gamma_1 v_1 \\ &\leq (u - \gamma_1 v_1)(x, y). \end{aligned}$$

By the maximum principle, for $|y| > R_2$,

$$u(x, y) - \gamma_1 v_1(x, y) \geq \min_{|y|=R_2} (u - \gamma_1 v_1)(x, y) \geq 0.$$

Thus, we have

$$\begin{aligned} u(x, y) &\geq \gamma_1 v_1(x, y) = \gamma_1 \exp(-\beta(|y| - R_2)) \\ &= \gamma_1 \exp(R_2 \beta) \exp(-\beta|y|) \\ &\geq \gamma_1 \exp(-\sqrt{1 + \lambda_1 + \delta}|y|) \quad \text{for } |z| \geq R_2. \end{aligned} \quad (4.2)$$

(ii) We know that positive numbers ε, c exist such that

$$0 \leq h(x, y) \leq c \exp(-\sqrt{1 + \lambda_1 + \delta}|y|) \quad \text{for any } (x, y) \in \Omega.$$

For $0 < \delta < 1 + \lambda_1$, by (4.2), there is $R_3 > R_2 > 0$ such that

$$\frac{\delta}{2} u(x, y) \geq h(x, y) \quad \text{for } |y| \geq R_3. \quad (4.3)$$

Since $\lim_{|y| \rightarrow \infty} u(x, y) = 0$, there exists $R_0 > R_3 > 0$ such that

$$1 - u^{p-2}(x, y) \geq 1 - \frac{\delta}{2} \quad \text{for } |y| \geq R_0. \quad (4.4)$$

Let $\gamma = \sqrt{1 + \lambda_1 - \delta}$ and $v_2(x, y) = v \phi_1(x) \exp(-\gamma(|y| - R))$, where $v = \max_{|y|=R} u(x, y) > 0$. Thus $\min_{|y|=R} (v_2 - u)(x, y) \geq 0$. By (4.3) and (4.4), for $|y| > R_0$,

$$\begin{aligned} \Delta(v_2 - u)(x, y) &= \left(\gamma^2 - \frac{\gamma(m-1)}{|y|}\right) v_2(x, y) - u + |u|^{p-2}u + h(x, y) \\ &\leq \gamma^2 v_2(x, y) - \left(1 - \frac{\delta}{2}\right) u(x, y) + h(x, y) \end{aligned}$$

$$\begin{aligned}
&= (1 + \lambda_1 - \delta)(v_2(x, y) - u(x, y)) - \frac{\delta}{2}u + h(x, y) \\
&\leq (1 + \lambda_1 - \delta)(v_2(x, y) - u(x, y)).
\end{aligned}$$

By the maximum principle, for $|y| > R_0$,

$$v_2(x, y) - u(x, y) \geq \min_{|y|=R_0} (v_2 - u)(x, y) \geq 0.$$

Thus, we have

$$\begin{aligned}
u(x, y) &\leq v_2(x, y) = v \exp(-\gamma(|y| - R)) = v \exp(R\gamma) \exp(-\gamma|y|) \\
&\leq \gamma_2 \exp(-\sqrt{1 + \lambda_1 - \delta}|y|) \quad \text{for } |y| \geq R.
\end{aligned}$$

This completes the proof. \square

By Lemma 4.2, there is an $R > 0$ such that $\omega \subset B^N(0; R) \cap \mathbf{A}$. For such R , let $\psi_R : \mathbf{A} \rightarrow [0, 1]$ be a C^∞ -function on \mathbf{A} such that $0 \leq \psi_R \leq 1$,

$$\psi_R(x, y) = \begin{cases} 1 & \text{for } |y| \geq R + 1, \\ 0 & \text{for } |y| \leq R. \end{cases}$$

For y_0 a unit vector in \mathbb{R}^l , we define

$$v_l(x, y) = \psi_R(x, y)w(x, y - ly_0).$$

Clearly, $v_l(x, y) \in H_0^1(\Omega)$.

Then we have the following lemmas.

Lemma 4.3.

- (i) $a(v_l) = b(v_l) + o(1)$ as $l \rightarrow \infty$;
- (ii) $J(v_l) = \alpha_0(\mathbf{A}) + o(1)$ as $l \rightarrow \infty$;
- (iii) $v_l \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $l \rightarrow \infty$.

Proof. See Wang [17, Lemma 30]. \square

Since Ω is noncontraction and in which there is no any ground state solution, we need more analyses.

Lemma 4.4. *There exists $l_0 > 0$ such that for $l \geq l_0$,*

$$\sup_{t \geq 0} J_h(u_0 + tv_l) < J_h(u_0) + \alpha_0(\mathbf{A}) \quad \text{uniformly in unit vector } y_0.$$

Proof. Our proof is almost the same as that in Hsu–Wang [13]. \square

For the Lusternik–Schnirelman category theory, see Ambrosetti [2] and Adachi–Tanaka [1, Lemma 2.5]. In the following, we take the idea of Adachi–Tanaka [1]. For $c \in \mathbb{R}$, we denote

$$[J_h \leq c] = \{u \in \mathbf{M}_h^- \mid J_h(u) \leq c \text{ and } u \geq 0\}.$$

We show for a sufficiently small $\sigma > 0$,

$$\text{cat}([J_h \leq \alpha_h(\Omega) + \alpha_0(\mathbf{A}) - \sigma]) \geq 2. \quad (4.5)$$

Let

$$A_1 = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{H^1}} t^- \left(\frac{u}{\|u\|_{H^1}} \right) > 1 \right\} \cup \{0\},$$

$$A_2 = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \frac{1}{\|u\|_{H^1}} t^- \left(\frac{u}{\|u\|_{H^1}} \right) < 1 \right\}.$$

Lemma 4.5. *We have the following results:*

- (i) $H_0^1(\Omega) \setminus \mathbf{M}_h^- = A_1 \cup A_2$;
- (ii) $\mathbf{M}_h^+ \subset A_1$;
- (iii) *There exist $t_0 > 1$ and $l_1 \geq l_0$ such that $u_0 + t_0 v_m \in A_2$ for each $l \geq l_1$, where l_0 is defined as in Lemma 4.4;*
- (iv) *There exist $s_l \in (0, 1)$ such that $u_0 + s_l t_0 v_l \in \mathbf{M}_h^-$ for each $l \geq l_1$;*
- (v) $\alpha_h^-(\Omega) < \alpha_h(\Omega) + \alpha_0(\mathbf{A})$.

Proof. Our proof is almost the same as that in Tarantello [16]. \square

By Lemma 4.5(iv), there exist $s_l \in (0, 1)$ such that $u_0 + s_l t_0 v_l \in \mathbf{M}_h^-$ for each $l \geq l_1$. For $l \geq l_1$, we define a map $F_l : S^{m-1} \rightarrow H_0^1(\Omega)$ by

$$F_l(y_0)(z) = u_0(z) + s_l t_0 v_l(z) \quad \text{for } y_0 \in S^{m-1}.$$

Then we have

Lemma 4.6. *There exists a sequence $\{\sigma_l\}$ in \mathbb{R}^+ such that*

$$F_l(S^{m-1}) \subset [J_h \leq \alpha_h(\Omega) + \alpha_0(\mathbf{A}) - \sigma_l].$$

Proof. By Lemma 4.5(iv) and Lemma 4.4, we have that for each $l \geq l_1$, $F_l(\bar{z}) = u_0 + s_l t_0 v_l \in \mathbf{M}_h^-$ and $J_h(F_l(\bar{z})) = J_h(u_0 + s_l t_0 v_l) \leq \alpha_h(\Omega) + \alpha_0(\mathbf{A}) - \sigma_l$, the conclusion holds. \square

For $c > 0$, we define

$$b_c(u) = \int_{\Omega} c|u|^p, \quad I_c(u) = \frac{1}{2}a(u) - \frac{1}{p}b_c(u),$$

$$\mathbf{M}_{I_c} = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle I'_c(u), u \rangle = 0\}.$$

Therefore, if $u \in \mathbf{M}_{I_c}$, then $a(u) = b_c(u)$, and $I_c(u) = (\frac{1}{2} - \frac{1}{p})b_c(u)$. Recall that there exist unique $t^- = t^-(u) > 0$ and $t^0 = t^0(u) > 0$ such that $t^-u \in \mathbf{M}_h^-$, $t^0u \in \mathbf{M}_0$, and $t^0(u) = (\frac{1}{b(u)})^{1/p-2}$. Similarly, we have

Lemma 4.7. *For each $u \in \Sigma$, we have the following results:*

- (i) *There exists a unique $t^c(u) > 0$ such that $t^c(u)u \in \mathbf{M}_{I_c}$ and*

$$\max_{t \geq 0} I_c(tu) = I_c(t^c(u)u) = \left(\frac{1}{2} - \frac{1}{p} \right) b_c(u)^{-\frac{2}{p-2}};$$

(ii) For $0 < \mu < 1$, there exists a $d_1(\mu) > 0$ such that for $\|h\|_{L^2} < d_1(\mu)$,

$$J_h(t^-u) \geq (1 - \mu)^{\frac{p}{p-2}} J_0(t^0u) - \frac{1}{2\mu} \|h\|_{L^2}^2.$$

Proof. (i) By elementary calculus.

(ii) For $0 < \mu < 1$, let $c = \frac{1}{1-\mu}$, $t^c > 0$ and $t^0 = t^0(u) > 0$ such that $t^c u \in \mathbf{M}_{I_c}$ and $t^0 u \in \mathbf{M}_0$. We have

$$\left| \int_{\Omega} h t^c u \, dz \right| \leq \|t^c u\|_{H^1} \|h\|_{L^2} \leq \frac{\mu}{2} \|t^c u\|_{H^1}^2 + \frac{1}{2\mu} \|h\|_{L^2}^2.$$

Then by part (i),

$$\begin{aligned} \sup_{t \geq 0} J_h(tu) &\geq J_h(t^c u) \geq \frac{1-\mu}{2} \|t^c u\|_{H^1}^2 - \frac{1}{p} b(t^c u) - \frac{1}{2\mu} \|h\|_{L^2}^2 \\ &= (1-\mu) \left[\frac{1}{2} \|t^c u\|_{H^1}^2 - \frac{1}{(1-\mu)p} \int_{\Omega} |t^c u|^p \right] - \frac{1}{2\mu} \|h\|_{L^2}^2 \\ &= (1-\mu) I_c(t^c u) - \frac{1}{2\mu} \|h\|_{L^2}^2 \\ &= (1-\mu)^{\frac{p}{p-2}} \left(\frac{1}{2} - \frac{1}{p} \right) b(u)^{-\frac{2}{p-2}} - \frac{1}{2\mu} \|h\|_{L^2}^2 \\ &= (1-\mu)^{\frac{p}{p-2}} J_0(t^0 u) - \frac{1}{2\mu} \|h\|_{L^2}^2 \\ &\geq (1-\mu)^{\frac{p}{p-2}} \alpha_0(\Omega) - \frac{1}{2\mu} \|h\|_{L^2}^2. \end{aligned}$$

For $\mu \in (0, 1)$, there exists a $d_1(\mu) > 0$ such that for $\|h\|_{L^2} < d_1(\mu)$,

$$\sup_{t \geq 0} J_h(tu) > 0.$$

By Lemma 3.2, there exists a $t^- = t^-(u) > 0$ such that $t^- u \in \mathbf{M}_h^-$ and

$$\sup_{t \geq 0} J_h(tu) = J_h(t^- u).$$

Thus, for $\|h\|_{L^2} < d_1(\mu)$,

$$J_h(t^- u) \geq (1 - \mu)^{\frac{p}{p-2}} J_0(t^1 u) - \frac{1}{2\mu} \|h\|_{L^2}^2.$$

This completes the proof. \square

Lemma 4.8. *There exists a positive number $d_0 < d_1(\mu)$ such that for $\|h\|_{L^2} < d_0$, we have*

$$\int_{\mathbf{A}} \frac{y}{|y|} (|\nabla u|^2 + u^2) \, dy \, dx \neq 0$$

for $u \in [J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})]$.

Proof. For $u \in [J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})]$, there exists a $t^0 > 0$ such that $t^0 u / \|u\|_{H^1} \in \mathbf{M}_0$. By Lemma 4.7(ii), we have for each $\mu \in (0, 1)$, there is $d_1(\mu) > 0$ such that $\|h\|_{L^2} < d_1(\mu)$ implies

$$J\left(\frac{t^1 u}{\|u\|_{H^1}}\right) \leq (1 - \mu)^{-\frac{p}{p-2}} \left(J_h(u) + \frac{1}{2\mu} \|h\|_{L^2}^2 \right), \quad (4.6)$$

where $\frac{t^- u}{\|u\|_{H^1}} = u \in \mathbf{M}_h^-$. Since $\alpha_h(\Omega) < 0$, we have $[J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})] \subset [J_h < \alpha_0(\mathbf{A})]$. Thus by (4.6), we have, for $u \in [J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})]$,

$$\begin{aligned} J\left(\frac{t^1 u}{\|u\|_{H^1}}\right) &\leq (1 - \mu)^{-\frac{p}{p-2}} \left(\alpha_0(\mathbf{A}) + \frac{1}{2\mu} \|h\|_{L^2}^2 \right) \\ &= \alpha_0(\mathbf{A}) + \epsilon(\mu) + (1 - \mu)^{-\frac{p}{p-2}} \frac{1}{2\mu} \|h\|_{L^2}^2, \end{aligned}$$

where $\epsilon(\mu) \rightarrow 0$ as $\mu \rightarrow 1$. Thus for $\delta_0 > 0$ in Lemma 2.8, there exist $\mu \in (0, 1)$ and $d_0 > 0$ such that for $\|h\|_{L^2} < d_0$, we have

$$J\left(\frac{t^1 u}{\|u\|_{H^1}}\right) \leq \alpha_0(\mathbf{A}) + \delta_0. \quad (4.7)$$

Since $t^0 u / \|u\|_{H^1} \in \mathbf{M}_0$, by Lemma 2.8 and (4.7) we have

$$\int_{\mathbf{A}} \frac{y}{|y|} \left(\left| \nabla \left(\frac{t^1 u}{\|u\|_{H^1}} \right) \right|^2 + \left(\frac{t^1 u}{\|u\|_{H^1}} \right)^2 \right) dy dx \neq 0,$$

or

$$\int_{\mathbf{A}} \frac{z}{|z|} (|\nabla u|^2 + (u)^2) dz \neq 0.$$

This completes the proof. \square

We hence define

$$G : [J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})] \rightarrow S^{m-1}$$

by

$$G(u) = \int_{\mathbf{A}} \frac{y}{|y|} (|\nabla u|^2 + |u|^2) dy dx \Big/ \left| \int_{\mathbf{A}} \frac{y}{|y|} (|\nabla u|^2 + |u|^2) dy dx \right|.$$

Then we have

Lemma 4.9. For $l \geq l_1$ and $\|h\|_{L^2} < d_0$, the map

$$G \circ F_l : S^{m-1} \rightarrow S^{m-1}$$

is homotopic to the identity.

Proof. Since $0 \leq h(x, y) \leq c \exp(-\sqrt{1 + \lambda_1 + \delta}|y|)$ for any $(x, y) \in \Omega$, then by regularities, we have $u_0, w \in C^{1,\theta}(\overline{\Omega}) \cap L^\infty(\Omega)$. We define

$$\zeta_l(\theta, y_0) : [0, 1] \times S^{m-1} \rightarrow S^{m-1}$$

by

$$\zeta_l(\theta, y_0) = \begin{cases} G((1-2\theta)F_l(y_0) + 2\theta\psi w(x, y - ly_0)) & \text{for } \theta \in [0, 1/2), \\ G(\psi w(x, y - \frac{l}{2(1-\theta)}y_0)) & \text{for } \theta \in [1/2, 1), \\ y_0 & \text{for } \theta = 1. \end{cases}$$

We have

(a) $\lim_{\theta \rightarrow 1^-} \zeta_l(\theta, y_0) = y_0$: Since

$$\begin{aligned} & \int_{\mathbf{A}} \frac{y}{|y|} \left(\left| \nabla \left[\psi w \left(x, y - \frac{l}{2(1-\theta)} y_0 \right) \right] \right|^2 + \left[\psi w \left(x, y - \frac{l}{2(1-\theta)} y_0 \right) \right]^2 \right) dy dx \\ &= \int_{\mathbf{A}} \frac{y + \frac{l}{2(1-\theta)} y_0}{|y + \frac{l}{2(1-\theta)} y_0|} (|\nabla w|^2 + w^2) dy dx \\ &= \left(\frac{2p}{p-2} \right) \alpha_0(\mathbf{A}) y_0 + o(1) \quad \text{as } \theta \rightarrow 1^-. \end{aligned}$$

(b) $\lim_{\theta \rightarrow \frac{1}{2}^-} \zeta_l(\theta, y_0) = G(\psi w(z - ly_0))$: We have

$$\|(1-2\theta)F_l(y_0) + 2\theta\psi w(x, y - ly_0)\|_{H^1} = \|w(x, y - ly_0)\|_{H^1} + o(1) \quad \text{as } \theta \rightarrow \frac{1}{2}^-.$$

By the continuity of G , we obtain $\lim_{\theta \rightarrow \frac{1}{2}^-} \zeta_l(\theta, y_0) = G(w(x, y - ly_0))$.

Thus, $\zeta_l(\theta, y_0) \in C([0, 1] \times S^{m-1}, S^{m-1})$ and

$$\zeta_l(0, y_0) = G(F_l(y_0)) \quad \text{for all } y_0 \in S^{m-1},$$

$$\zeta_l(1, y_0) = y_0 \quad \text{for all } y_0 \in S^{m-1},$$

provided $l \geq l_1$ and $\|h\|_{L^2} < d_0$. \square

Thus we have

Lemma 4.10. $J_h(u)$ has at least two critical points in

$$[J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})].$$

Proof. Applying Adachi–Tanaka [1, Lemma 2.5], Proposition 3.4 and Lemma 4.9, we have for sufficiently large $l \geq l_1$ and $\|h\|_{L^2} < d_0$,

$$\text{cat}([J_h \leq \alpha_h(\Omega) + \alpha_0(\mathbf{A}) - \sigma_l]) \geq 2.$$

By Ambrosetti [2, Theorem 2.3] and Lemma 4.5(v), $J_h(u)$ has at least two critical points in $[J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})]$. \square

We can now complete the proof of Theorem 1.1. We need to show that

$$J_h(u_0) = \alpha_h(\Omega) < \alpha_h^-(\Omega) = J_h(u^0)$$

for each critical point u^0 in $[J_h < \alpha_h(\Omega) + \alpha_0(\mathbf{A})] \subset \mathbf{M}_h^-$. Otherwise, assume that $J_h(u^0) = \alpha_h^-(\Omega) = J_h(u_0) = \alpha_h(\Omega)$. Since $\int_{\Omega} h u^0 > 0$, by Lemma 3.2, there exists $t^+(u^0) > 0$ such that $t^+(u^0) u^0 \in \mathbf{M}_h^+$ and

$$\alpha_h^+(\Omega) \leq J_h(t^+(u^0) u^0) < J_h(u^0) = \alpha_h(\Omega) = \alpha_h^-(\Omega),$$

which contradicts to $\alpha_h^+(\Omega) = \alpha_h(\Omega)$ in Theorem 3.5. Therefore, we have that Eq. (1.1) in $\mathbf{A} \setminus \overline{D}$ has at least three nonnegative solutions. Moreover, since $h \not\equiv 0$, then by maximum principle, Eq. (1.1) in $\mathbf{A} \setminus \overline{D}$ has at least three positive solutions. \square

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